Emergent photons and transitions in the O(3) sigma model with hedgehog suppression

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We study the effect of hedgehog suppression in the O(3) sigma model in $D=2+1$. We show via Monte Carlo simulations that the sigma model can be disordered while effectively forbidding these point topological defects. The resulting paramagnetic state has gauge charged matter with half-integer spin (spinons) and also an emergent gauge field (photons), whose existence is explicitly demonstrated. Hence, this is an explicit realization of fractionalization in a model with global SU(2) symmetry. The zero-temperature ordering transition from this phase is found to be continuous but distinct from the regular Heisenberg ordering transition. We propose that these phases and this phase transition are captured by the noncompact CP^1 model, which contains a pair of bosonic fields coupled to a noncompact U(1) gauge field. Direct simulation of the transition in this model yields critical exponents that support this claim. The easy-plane limit of this model also displays a continuous zero temperature ordering transition, which has the remarkable property of being self-dual. The presence of emergent gauge charge and hence Coulomb interactions is evidenced by the presence of a finite temperature Kosterlitz-Thouless transition associated with the thermal ionization of the gauge charged spinons. Generalization to higher dimensions and the effects of nonzero hedgehog fugacity are discussed.

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I. INTRODUCTION

Since the initial proposal of fractionalization (phases where the elementary excitations are fractions of the electron) as the underlying explanation for the unusual properties of the cuprate superconductors,[1] there has been much progress in the theoretical understanding of these phases. Whether such fractionalized phases in the absence of magnetic fields in spatial dimension greater than 1 are actually realized in any experimental system is a matter of current debate. Fractionalized phases can exhibit properties that are strikingly different from conventional phases of matter, hence they are attractive candidates for modeling strongly correlated systems that exhibit anomalous behaviour. However, unambiguous experimental evidence for the presence of such phases in any experimental system is still lacking. In part this may be because the correlations in such phases are subtle, and hence definitive experimental signatures are hard to devise. This provides a strong motivation to seek a deeper understanding of these phases. Furthermore, fractionalized states have been proposed as a means to build quantum bits that are inherently robust against decoherence.[2]

An important theoretical development has been the discovery of a number of microscopic models[3–11] that can be shown to exhibit this exotic physics. Although these microscopic models are defined in terms of bosons (or spins) on a lattice with short ranged interactions, fractional excitations that are charged under an emergent gauge field, as well as excitations of this emergent gauge field are obtained on solving these models. However to date, there have been no microscopic models available with the full SU(2) spin rotation symmetry. Indeed it is important to verify that the additional constraints imposed by spin rotation symmetry do not exclude the possibility of fractionalization. Here we will describe a model that possesses the full spin rotation symmetry, but can be explicitly shown to exhibit fractionalization and possess an emergent gauge field in the deconfined phase. Moreover this model is found to have a quantum critical point with full spin rotation symmetry, but which is distinct from the Heisenberg transition. The properties of the fractionalized phase and the transition, as well as various deformations on the model, will be studied in detail in this paper.

Most of the earlier work that constructed models exhibiting fractionalization engineered the energetics so as to select a low energy manifold. Constraining states to lie in this manifold introduces the gauge fields, which then need to be in the deconfined phase for fractionalization to occur. Here we will rely on a different route to fractionalization, which may be described as fractionalization from defect suppression. Indeed this approach is closer in spirit to Ref. 12, where the Z_2 fractionalized state was regarded as a quantum disordered superconductor, obtained by proliferating even winding-number vortices while suppressing vortices of odd winding number.

Here we will mainly be concerned with the O(3) sigma model in $D=2+1$. This model consists of O(3) quantum rotors represented by unit three-vectors ("spins"), defined on the sites of a two dimensional spatial lattice. Neighboring rotors are coupled via a ferromagnetic interaction. By the usual quantum to classical statistical physics mapping, the ground-state properties of this model can be conveniently mapped onto the physics of the Heisenberg model at finite temperature in three dimensions. Clearly, there exist point topological defects in the three-dimensional Heisenberg model that carry an integer topological charge, which simply correspond to hedgehog configurations of the spins. In terms of the quantum model, these are events in space-time (instantons) which change the skyrmion number of the system. We now ask the question: Is it possible to disorder the three dimensional Heisenberg model in the effective absence of the hedgehog defects? This has been a long standing issue, discussed in several works, for example Refs. 13 and 14, but had not been conclusively settled. Here we will present fresh results from Monte Carlo simulations and arguments that convincingly demonstrate that the answer to this question is yes.

Having established this, we will ask the question, what is the nature of this hedgehog-free paramagnetic phase P^*? Although the spin-spin correlations are short ranged, the absence of hedgehog fluctuations lead to a "hidden order" in this phase. Indeed, it will be proposed that the physics of the hedgehog-free model is captured by what we will call the non compact CP^1 (NCCP^1) (Ref. 15) model in $D=2+1$. This model consists of a doublet of bosonic fields ("spinons") that transforms as a spinor under spin rotations, coupled to a noncompact U(1) gauge field ("photon"). In this representation, the hedgehogs correspond to the monopoles of the U(1) gauge field, and eliminating hedgehogs leads to the noncompactness of the gauge field. The NCCP^1 model has two obvious phases, one where the spinons are condensed which is the ferromagnetic phase, and the other where the spinons are gapped, which corresponds to an exotic paramagnet, with a gapless photon excitation. To verify that the paramagnetic phase P^* obtained by the hedgehog-free disordering of the O(3) sigma model is indeed the same phase as the paramagnet in the NC CP^1model, we use the Monte Carlo method to measure the correlations of the spin chirality, which is roughly $\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)$ for a triangular face with three spins $\mathbf{n}_{1,2,3}$. This should be equivalent to the flux correlations of the noncompact gauge theory. Indeed the expected long-range correlation functions with the very characteristic dipolar form are found. Furthermore, this correspondence implies that the ordering transition of hedgehog-free O(3) model is not in the Heisenberg universality class, but in the same universality class as the ordering transition of the NC CP^1 model. This may be checked by comparing the universal critical exponents in the two models. Indeed, we find that the hedgehog-free O(3) model undergoes a continuous ordering transition with critical exponents that are clearly distinct from the Heisenberg exponents, but which are consistent with the exponents obtained in direct Monte Carlo simulation of the transition in the NCCP^1model. This provides further evidence that the NCCP^1 model captures the physics of the hedgehog-free O(3) model. These exponents also turn out to be consistent with those obtained from an earlier attempt to disorder the hedgehog-free O(3) model.[14]

Given the central role played by the NCCP^1 model, we also present analytical results on some of the striking properties of this model. First, we consider the easy-plane deformation of the NCCP^1 model when the full spin rotation symmetry is broken down to U(1) by the presence of easy-plane anisotropy. This model is found to have the amazing property that under the standard duality transformation, it maps back onto itself. In particular the zero-temperature ordering transition is found to be self-dual. Second, we consider the effect of finite temperature in the NCCP^1 model. We argue that there is a thermal Kosterlitz-Thouless phase transition out of the P^* phase. This can be understood as an ionization transition of the logarithmically interacting spinons. The logarithmic interaction is of course the Coulomb potential in two spatial dimensions and hence the presence of such a transition is proof of the existence of gauge charged particles.

We now briefly comment on the relation of the present paper to earlier relevant work. Lau and Dasgupta,[13] considered the O(3) model in three Euclidean dimensions on a cubic lattice and applied complete monopole suppression at every cube of the lattice. This strong constraint led to the model always being in the ferromagnetic phase, and the exotic paramagnet was not uncovered in that work. Subsequently, in an important extension, Kamal and Murthy[14] allowed for a more flexible definition of the no monopole constraint by allowing monopole-antimonopole pair fluctuations if they occurred on neighboring cubes. In this way, they were able to obtain a disordered phase, and also found a continuous ordering transition with non-Heisenberg exponents. However, as pointed out in that work itself, there are unsatisfactory features of this algorithm when closely spaced monopoles occur. The problem arises in the presence of loops of monopole and antimonopoles, where combining them in pairs is ambiguous. This allows for Monte Carlo moves that annihilate a monopole and an antimonopole belonging to different pairs, since the remaining monopoles and antimonopoles on the loop can be "repaired." This requires making a time consuming nonlocal check for repairing, every time such an event is generated. Consequently, Kamal and Murthy had to resort to the approximation of making only local checks of repairing. The algorithm then also has the undesirable property that a given configuration of spins may be allowed or not allowed depending on the history of how it is generated. Here we will adopt a more restrictive condition—that monopole-antimonopole pairs are allowed only if they are *isolated*—to completely circumvent these problems. An improved procedure for defining the monopole number allows us to easily work with more complicated lattices. This flexibility will prove very useful in obtaining the disordered phase. Furthermore, our identification of the NCCP^1 model to describe this physics allows us to bring a whole series of tests to bear on the P^* phase and the transition, which was not done previously in the absence of such an understanding.

In this paper we will be completely suppressing the free hedgehog defects, i.e., setting their fugacity to zero. We now briefly discuss the effect of a finite fugacity, and possible relevance of the physics described here to systems that may be realized in nature. It is well known that for pure U(1) gauge theories in $D=2+1$, introduction of monopoles leads to confinement.[16] This result also implies that the P^* phase is unstable to the introduction of a nonzero hedgehog (or monopole) fugacity. However, if this fugacity is small to begin with, deconfinement physics would play an important role in the finite temperature or short wavelength properties of the system.

While a finite monopole fugacity necessarily destroys the zero-temperature P^* phase, its effect on the phase transition, where critical gauge charged bosons are present which act to hinder monopole tunneling, is less obvious. In fact, in the sigma model description of the spin-half antiferromagnet on the square lattice, Berry phase effects[17] lead to a quadrupling of the monopoles,[18] which are then more likely to be irrelevant as compared to single monopole insertions. If monopoles are then irrelevant at the critical point, the transition could still be controlled by the monopole suppressed O(3) critical point (or equally the critical NCCP^1 model). However, monopole relevance in the adjoining P^* phase implies that this transition is sandwiched between two conventional (not deconfined) phases. This dangerously irrelevant monopoles scenario is advocated in a forthcoming paper,[19] which argues that it is possible to have a continuous transition between a Néel and valence bond solid (VBS) state for the square lattice spin-half antiferromagnet, which is in the same universality class as the hedgehog free O(3) transition (i.e., the critical NCCP^1 theory) studied here. In fact, Ref. 19 argues that the easy plane version of this transition may already have been seen in the numerical experiments of Ref. 20, where there appears to be a continuous transition between a spin ordered state and a VBS in a square lattice spin-half model with easy plane anisotropy. Such a direct transition is very natural in the dangerously irrelevant monopoles scenario, and would be controlled by the critical NCCP^1 model with easy plane anisotropy, which is also studied in this paper. Thus, the transitions of the hedgehog suppressed O(3) model, in both the isotropic and easy plane limits, could be realized in these situations even without explicit hedgehog suppression, and might potentially be seen directly in nature. Finally, we note that in $D=3+1$ a deconfined phase, with photons and gapped spin-half particles, can in principle exist even with a nonzero hedgehog fugacity.

The layout of this paper is as follows. In Sec. II we study the O(3) sigma model with hedgehog suppression. We begin by describing the particular lattice geometry and hedgehog suppression scheme used in the Monte Carlo calculations. We then present Monte Carlo results that show the presence of a spin disordered phase in our hedgehog suppressed model. We argue that the physics of the hedgehog suppressed sigma model is captured by the NCCP^1 model, which implies the presence of photons in this disordered phase. This leads to the prediction that spin chirality correlations in the disordered phase take on a very particular long ranged form which is tested in the Monte Carlo calculations. Next, we turn to the universal properties of the ordering transition in this hedgehog suppressed model and find exponents that are distinct from the Heisenberg exponents. These are then compared with exponents calculated from directly simulating the NCCP^1 model. In Sec. III, we consider various deformations of the NCCP^1 model that correspond to an easy plane anisotropy, a Zeeman field, and the effect of finite temperature. In particular we prove the remarkable self-duality of the easy plane model. Finally, in Sec. IV we briefly discuss possible higher-dimensional extensions of this physics.

II. THE HEDGEHOG FREE O(3) SIGMA MODEL

A. The model

We perform Monte Carlo simulations of the three dimensional classical O(3) sigma model with hedgehog suppression. The lattice that we consider is a decorated cubic lattice as shown in Fig. 1(a), with unit vectors $\mathbf{n}_i$ at the vertices and edge centers of the cubic lattice. As described below, monopole numbers are associated with the centers of each of these

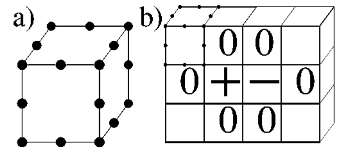

FIG. 1. Decorated cubic lattice used in the simulations. Spins live on the lattice points shown, and the monopole number is defined within each cube. (b) The only spin configurations accepted in the simulation are those that are either hedgehog free, or have hedgehogs that can be paired uniquely into isolated nearest-neighbor hedgehog-antihedgehog pairs. A schematic depiction of such a pairing is shown here in a vertical section through an isolated pair.

cubes. This choice allows for more spin fluctuations when hedgehog suppression is applied than the simple cubic geometry of Refs. 13 and 14. Neighboring spins are coupled via the usual ferromagnetic Heisenberg interaction and hence allowed states are weighed with the factor e^{-E}, with the energy function E given by

$$E = -J\sum_{\langle ij\rangle} \mathbf{n}_i \cdot \mathbf{n}_j. \tag{1}$$

In order to define the monopole number in each cube, we follow Ref. 21 and first introduce an auxiliary variable, the gauge potential $\mathcal{A}_{ij}$ between any pair of neighboring sites with spin orientations $\mathbf{n}_i, \mathbf{n}_j$. This is defined by introducing an arbitrary reference vector $\mathbf{n}_*$ and forming the spherical triangle $(\mathbf{n}_*, \mathbf{n}_i, \mathbf{n}_j)$. The edges of a spherical triangle are of course segments of great circles. If the solid angle subtended by this spherical triangle is $\Omega[\mathbf{n}_*, \mathbf{n}_i, \mathbf{n}_j]$, then we define

$$\begin{aligned} e^{iA_{ij}} &= e^{(i/2)\Omega[\mathbf{n}_*,\mathbf{n}_i,\mathbf{n}_j]} \\ &= \frac{1 + \mathbf{n}_* \cdot \mathbf{n}_i + \mathbf{n}_* \cdot \mathbf{n}_j + \mathbf{n}_i \cdot \mathbf{n}_j + i\mathbf{n}_* \cdot (\mathbf{n}_i \times \mathbf{n}_j)}{\sqrt{2(1 + \mathbf{n}_* \cdot \mathbf{n}_i)(1 + \mathbf{n}_* \cdot \mathbf{n}_j)(1 + \mathbf{n}_i \cdot \mathbf{n}_j)}}. \end{aligned} \tag{2}$$

A different choice of the reference vector, n'_*, only leads to a 'gauge' transformation of $\mathcal{A}$: $\mathcal{A}_{ij} \to \mathcal{A}_{ij} + \chi_i - \chi_j$, where $\chi_i = \frac{1}{2}\Omega[\mathbf{n}'_*, \mathbf{n}_*, \mathbf{n}_i]$, etc. Thus, gauge invariant quantities are independent of the choice of the reference vector. Note also that $e^{i\mathcal{A}_{ij}} = e^{-i\mathcal{A}_{ji}}$ and these gauge fields are only defined modulo 2π. Thus we have defined a compact gauge field in terms of the spins. We then define a flux $F_\square$ on every face bounded by the sites $(1,2,\ldots,n,1)$:

$$e^{iF} = e^{i(\mathcal{A}_{12}+\mathcal{A}_{23}+\cdots+\mathcal{A}_{n1})} \tag{3}$$

with $F_\square \in (-\pi, \pi]$. Clearly, the flux is gauge invariant and hence independent of the choice of reference vector $\mathbf{n}_*$. The physical meaning of the flux is most readily appreciated by considering a triangular face with spins $\mathbf{n}_{1,2,3}$, where it is approximately the spin chirality

$$\sin F_\Delta \sim \mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3). \tag{4}$$

The hedgehog number k enclosed in a given volume is then the net flux out of this volume $\Sigma F_\square = 2\pi k$, which is guaranteed to be an integer from the previous definitions. Note that the hedgehog number is simply some function of the spins on a given cube. This definition is identical to the traditionally used definition of hedgehog number for volumes that are bounded by triangular faces. For more complicated geometries (like the one employed in this work) however it is a much more natural and powerful definition, since it does not rely on an arbitrary triangulation of the faces and can be quickly computed.

We now consider whether a disordered phase may be obtained while suppressing the hedgehog configurations. This will favour the ferromagnetic state which is clearly free of hedgehogs; indeed with full hedgehog suppression on the simple cubic lattice[13] an ordered phase was found even at zero spin coupling. In the decorated lattice shown in Fig. 1(a), full hedgehog suppression in each cube seems to give rise to a disordered state for small values of the spin coupling J. However, in order to open a larger window of disordered phase, and obtain more solid evidence of disorder in the system sizes available, we will allow for hedgehog-antihedgehog fluctuations on nearest neighbour (face sharing) cubes. In contrast to Ref. 14, we will only allow for configurations with isolated hedgehog-antihedgehog pairs; in other words, if a cube contains a hedgehog of strength q, it must contain a nearest-neighbor cube with a hedgehog of strength $-q$ and no hedgehogs in all other nearest-neighbor cubes. This is shown schematically in Fig. 1(b) and gives an unambiguous prescription for combining the hedgehog and antihedgehogs into isolated, neutral pairs, and allows us to avoid altogether the problems in the work of Ref. 14, where such an isolation of pairs was not demanded.

To summarize, the statistical ensemble is defined as follows. For each spin configuration, we determine the hedgehog numbers associated with each cube of the lattice. If this sample clears the constraint of no free hedgehogs (we mentioned two versions of this constraint, full suppression constraint and the isolated neutral pairs constraint), then this configuration is allowed in the ensemble and is weighted with a relative probability determined by the energy function (1).

We simulate this ensemble[22] using single spin Metropolis updates in the restricted configuration space. The data presented below is taken for 20 000–200 000 Metropolis per spin.

B. The disordered phase

We now discuss the results of the Monte Carlo simulation with hedgehog suppression. First, in the absence of any hedgehog suppression, the system is found to have the usual Heisenberg ordering transition at $J_{c,\mathrm{Heis}} \approx 1.7$. Implementing hedgehog suppression that only allows neutral, isolated pairs of hedgehogs to occur, gives a smaller but still sizeable region $0 \leqslant J < 0.7$ over which the system remains magnetically disordered. This can be seen in Fig. 2 where the magnetization per spin m is plotted for varying system sizes with linear dimension $L = 6, 8, 12, 16$ (the total number of spins is $N_{\mathrm{spin}} = 4L^3$). The magnetization per spin is seen to approach zero with increasing system size, for small enough values of J. A more convincing demonstration is made in the inset, where we plot the product of m and the square root of the total number of spins. For disordered spins, the average magnetization per spin is expected to decrease as $N_{\mathrm{spin}}^{-1/2}$. Indeed, as seen in the figure, this situation is realized at least for $J < 0.5$.

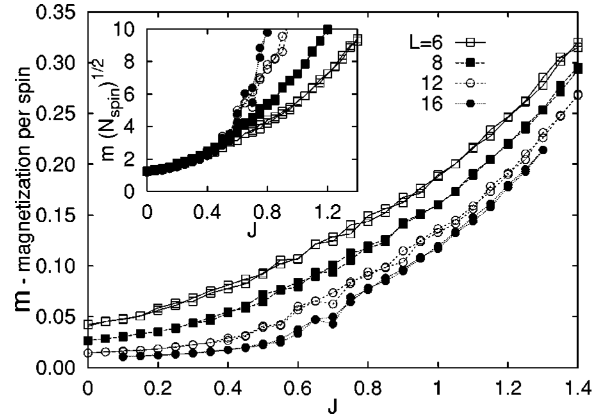

FIG. 2. Magnetization per spin $m = \langle |\mathbf{M}| \rangle / N_{\mathrm{spin}}$, with $\mathbf{M} = \Sigma_i \mathbf{n}_i$, as a function of J for different system sizes (we show the data for both sweep directions). The inset shows the product $mN_{\mathrm{spin}}^{1/2}$; in the magnetically disordered phase, we expect the measured $\langle |\mathbf{M}| \rangle \sim N_{\mathrm{spin}}^{1/2}$ (for completely uncorrelated spins, the numerical coefficient is close to 1).

One may nevertheless worry if there is some other spin order, such as antiferromagnetic or spiral order, that is not detected by the above zero-momentum magnetization. The most direct evidence against any magnetic order is obtained from the spin-spin correlation, which is found to be ferromagnetic throughout and rather short ranged. For $J = 0$ this is shown in Fig. 3, and the spin correlation indeed decays very quickly, with the correlation length of order one-half lattice spacing. The above does not mean that the spins are completely uncorrelated, rather that their correlation is more subtle as we will see below.

This completes the evidence for the presence of a magnetically disordered phase P^* with suppressed hedgehogs. We now investigate the nature of this paramagnetic phase.

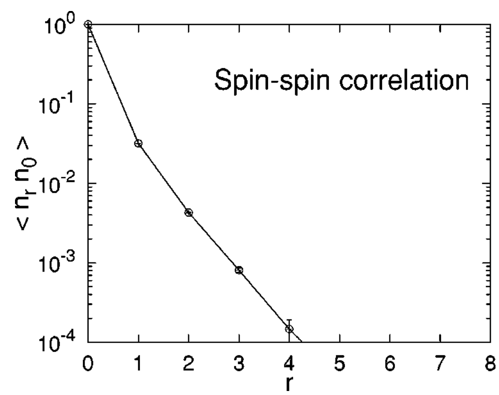

FIG. 3. Spin-spin correlations for $J = 0.0$, measured for spins at the vertices of the cubic lattice separated by a distance r along the $\hat{z}$ direction. The system size is $L = 16$, so the measurements are done for $r \leqslant 8$. Note the logarithmic scale for the vertical axis; the lower cutoff is roughly the limit of what can be reliably measured in our Monte Carlo simulation.

C. Emergent photons in P^*

We argue below that the P^* phase of the hedgehog suppressed O(3) model is distinct from the regular paramagnetic phase P in the model without such suppression. The sharp distinction arises from the fact that P^* contains a low-energy photon excitation. While this is best understood by rewriting the O(3) sigma model in the CP^1 representation, we first provide a heuristic argument for why such a low energy excitation may appear before passing to this more complete explanation.

At any given time slice, the spin configuration (now of spins in a plane) can be given a skyrmion number. It is easily seen that hedgehog events change the skyrmion number by unity. Therefore, suppressing hedgehogs implies that the skyrmion number is a conserved quantity. Thus, if j_0 is the skyrmion density and $j_{1,2}$ are skyrmion currents, they satisfy the conservation law $\partial_\mu j_\mu=0$ with $\mu=0, 1, 2$. This condition may be solved by writing $j_\mu=\epsilon_{\mu\nu\sigma}\partial_\nu a_\sigma$, in which the skyrmion current is identified with the flux of a U(1) gauge field. A natural dynamics would then be given by a Lagrangian $\mathcal{L}=j_\mu j_\mu$, which would give rise to a linearly dispersing photon.

To gain further insight into the nature of the P^* phase, we use the CP^1 representation of the O(3) sigma model. It is well known that the pure O(3) sigma model (no hedgehog suppression) can be rewritten in terms of a pair of complex bosonic fields $\mathbf{z}=(z_1\ z_2)^T$ that is minimally coupled to a compact gauge field. The fields $\mathbf{z}$ transform as spinors under spin rotations, and have unit magnitude $\mathbf{z}^\dagger\cdot\mathbf{z}=1$. The spin vector is given by the bilinear $\mathbf{n}=\mathbf{z}^\dagger\cdot\sigma\cdot\mathbf{z}$ (where σ are the Pauli matrices), and the flux of the gauge field corresponds to the skyrmion density of the original spin variables. Compactness of the gauge field implies the existence of monopoles, which act as sources or sinks of the gauge flux. These are then to be identified with the hedgehogs which change the skyrmion number when they occur. Clearly, this CP^1 model has two phases, one where the $\mathbf{z}$ particles are "condensed" which is the ferromagnetic phase (since the gauge neutral unit vectors $\mathbf{n}$ acquire an expectation value), and another where the $\mathbf{z}$ particles are gapped. The gapped phase is essentially equivalent, at low energies, to a pure compact gauge theory which is known to be confining in $D=2+1$.[16] This we associate with the regular paramagnetic phase.

We now turn to a description of the phases with full hedgehog suppression within the CP^1 representation. Indeed, given the identification of the hedgehog defects with the monopoles of the CP^1 theory, hedgehog suppression implies monopole suppression. This is most directly implemented by passing to a noncompact gauge field which is free of monoples. The distinction between a compact and a noncompact gauge theory cannot be overemphasized here, since it underlies all the new physics obtained in this work. The Euclidean action for the noncompact CP^1 model on a lattice is given by

$$S_{\mathrm{NC}CP^1}=-\frac{\mathcal{J}}{2}\sum_{r,\mu}(z_r^\dagger z_{r+\hat{\mu}}e^{ia_{r\mu}}+\text{c.c.})+\frac{K}{2}\sum_{\square}(\Delta\times\boldsymbol{a})^2, \quad (5)$$

where the lattice curl is the sum of the gauge potentials around a plaquette. This NCCP^1 model has two obvious phases—first, a phase where the $\mathbf{z}$ particles are "condensed" which is the ferromagnetic phase.[23] Second, a phase where the $\mathbf{z}$ particles are gapped—this we identify with the paramagnetic phase P^*. However, the noncompact nature of the gauge field implies that there will be a low energy photon excitation in this phase. Indeed, within the NCCP^1 model, the asymptotic correlation of the flux $C_{\mu\nu}=\langle F_\mu(r)F_\nu(0)\rangle$ in this phase is simply governed by the free propagation of the photon which leads to the characteristic dipolar form

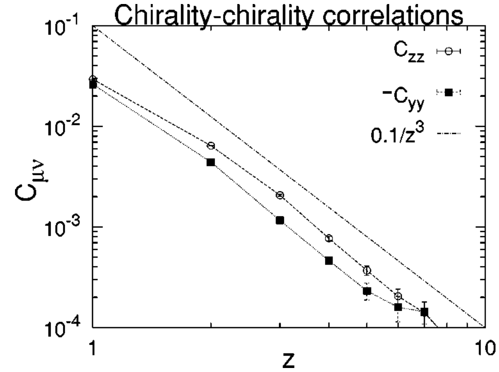

FIG. 4. Chirality-chirality (flux) correlations measured along the $\hat{z}$ direction for the same system as in Fig. 3. Note the logarithmic scale for both axes. We also show a $\sim 1/z^3$ line to indicate the observed power law falloff.

$$C_{\mu\nu}(r)\sim\frac{3r_\mu r_\nu-\delta_{\mu\nu}r^2}{r^5}. \quad (6)$$

In particular, for two points separated along the $\hat{z}$ direction, we would expect

$$C_{zz}(z)\approx\frac{2B}{z^3}, \quad C_{yy}(z)\approx-\frac{B}{z^3}, \quad (7)$$

where B is some numerical coefficient.

This prediction of the emergence of a photon in the P^* phase may be readily checked by using the definition (2), (3) of the flux in terms of the spins on a face (the spin chirality), and studying its correlations

$$C_{\mu\nu}(r)\equiv\langle\sin F_\mu(r)\sin F_\nu(0)\rangle \quad (8)$$

in the hedgehog suppressed sigma model (as usual, F_μ is the flux through a face perpendicular to $\hat{\boldsymbol{\mu}}$). The results are shown in Fig. 4 which was taken deep in the P^* phase with $J=0$. This figure shows chirality-chirality correlations for points separated along the $\hat{z}$ direction and corresponds to the prediction in Eq. (7). Indeed the expected $1/r^3$ falloff is reproduced, as well as the sign of correlations and their approximate relative magnitude. [For the data in Fig. 4, we find $C_{zz}(z)\approx-1.7\ C_{yy}(z)$; the slight discrepancy is most likely because the scaling regime is not quite reached for the separation in several lattice spacings.] Thus, although the spin-spin correlation function is short ranged in P^*, the spin chirality correlations have long-ranged power law forms. This is a result of the hidden internal order present in the system that arises from the suppression of hedgehog defects. It is this internal order that gives rise to the coherent photon excita-

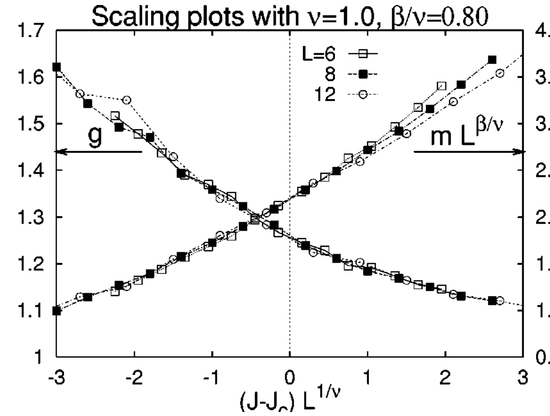

FIG. 5. Finite-size scaling plots for the cumulant ratio (left vertical axis) and magnetization (right axis), corresponding to the scaling forms (10) and (12). We used $J_c=0.725$, $\nu=1.0$, and $\beta/\nu=0.80$; the range of the horizontal axis corresponds roughly to $J\in[0.40,1.05]$ for $L=8$ (compare with Fig. 2).

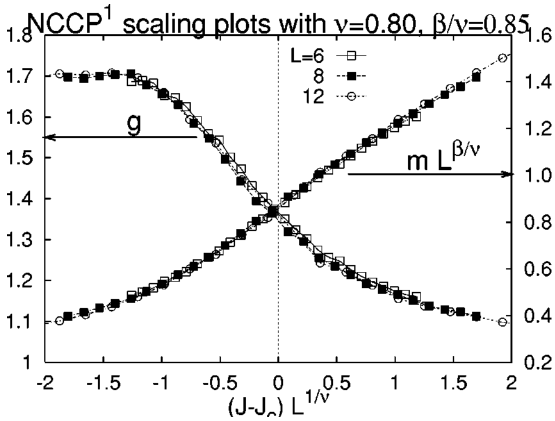

FIG. 6. Finite-size scaling study of the ordering transition in the NC CP^1model Eq. (5). The system is at $K=0.6$ and the transition from the P^* to the ferromagnetically ordered phase is observed at $J_c=1.255\pm0.02$; the measured order parameter is defined from the field $\mathbf{n}_r=\mathbf{z}_r^\dagger\sigma\mathbf{z}_r$. The scaling analysis is similar to Fig. 5(a). The exhibited plots are for $\nu=0.80$, $\beta/\nu=0.85$; the horizontal range corresponds to $J\in[1.10,1.40]$ for the $L=8$ system.

tion. (We also note that this topological order survives in a small applied Zeeman field $E\to E-h\Sigma_i n_i^z$, which supports our claim for the gapped spinons— see Sec. III B below and Ref. 25.)

D. The transition

We now study details of the ordering transition in the hedgehog suppressed O(3) model. We use standard finite size scaling analysis in order to estimate the corresponding critical indices. First, to find the critical point with good accuracy, we measure the cumulant ratio (Binder ratio)

$$g=\frac{\langle\mathbf{M}^4\rangle}{\langle\mathbf{M}^2\rangle^2}. \tag{9}$$

It is expected to have the finite size scaling form

$$g(J,L)=g(\delta L^{1/\nu}), \tag{10}$$

where $\delta\equiv J-J_c$ is the deviation from the critical point. In particular, the curves $g(J,L)$ for different fixed L all cross near the critical J_c, which we estimate to be $J_c=0.725\pm0.025$. Using the above scaling form, we also estimate the correlation length exponent

$$\nu=1.0\pm0.2 \ \ (\text{hedgehog suppressed O3}). \tag{11}$$

The corresponding scaling plot is shown in Fig. 5.

To estimate the exponent β, we study the finite size scaling of the magnetization per spin

$$m(J,L)=L^{-\beta/\nu}f(\delta L^{1/\nu}). \tag{12}$$

Our best scaling of the data is also shown in Fig. 5, and we find

$$\beta/\nu=0.80\pm0.05 \ \ (\text{hedgehog suppressed O3}). \tag{13}$$

We remark that these exponents are consistent with the exponents obtained in the study by Kamal and Murthy[14] on the same model but using a different monopole suppression scheme. These exponents are clearly different from the three-dimensional Heisenberg exponents which are accurately known to be[24] $\nu_{\text{Heis}}=0.705\pm0.003$ and $(\beta/\nu)_{\text{Heis}}=0.517\pm0.002$.

The specific heat exponent $\alpha=2-d\nu$ is expected to be negative for this transition (d is the space-time dimensionality) $\alpha\approx-1.0$, hence a cusp singularity is expected here. Although such singularities are harder to detect than a divergence, we nevertheless look for it in the Monte Carlo study of the hedgehog suppressed model. It is found, however, that the specific heat remains completely featureless across the transition; the reason for this behavior is unclear.

An important check on the above results in the hedgehog free O(3) sigma model is provided by comparing with direct simulations on the noncompact CP^1 model given by the Euclidean action (5). A complete numerical phase diagram of this model is reported elsewhere,[25] here we focus on the transition between the P^* and F phases and comparing the exponents with those obtained from the hedgehog free O(3) sigma model. Indeed we find that this transition is second order with exponents

$$\nu=0.8\pm0.1,\ \ \beta/\nu=0.85\pm0.05\ \ (\text{NC}CP^1), \tag{14}$$

where the exponent β of course describes the onset of ordering in the gauge neutral field $\mathbf{n}=\mathbf{z}^\dagger\cdot\sigma\cdot\mathbf{z}$. The corresponding scaling plots for this transition are shown in Fig. 6. Clearly, these are consistent with the exponents we obtained earlier for the hedgehog free O(3) sigma model, which leads us to conclude that this transition is indeed distinct from the Heisenberg ordering transition and is instead described by the deconfined to Higgs transition in the NCCP^1 model. The remaining difference between our best estimates for the critical indices [and also between the apparent values of the Binder cumulant at the critical points $g(J_c)\approx1.35$ for the NCCP^1 transition, and $g(J_c)\approx1.25$ for the hedgehog suppressed O(3) transition] we attribute to the small system sizes considered. For example, for the sizes studied here, the raw data (not shown here) Binder cumulant crossing in the NCCP^1 model has a clear downward trend with increasing system sizes, while it changes more slowly in the hedgehog suppressed O(3) model; this would narrow the difference on going to larger system sizes.

Finally let us note an important physical distinction between this transition and the Heisenberg transition, which is brought out by comparing the η exponents. This exponent is related to the anomalous dimension of the **n** field and is given by $\eta=2-d+2(\beta/\nu)$. For the Heisenberg transition, this exponent is very small $\eta=0.033\pm0.004$. However, for the hedgehog suppressed O(3) transition we find this exponent to be fairly large $\eta\approx0.6$ (and $\eta\approx0.7$ from direct simulations of the NCCP^1 model). A large η is to be expected if the magnon field **n** can decay into two deconfined spinons. Indeed, in the limit of noninteracting spinons, this exponent is expected to approach unity. Hence, the large η obtained at this transition is to be expected on physical grounds.

III. DEFORMATIONS OF THE NCCP^1 MODEL: EASY PLANE, ZEEMAN FIELD, AND EFFECTS OF FINITE TEMPERATURE

In this section we consider various deformations of the NC CP^1 model, and the effect of finite temperatures. The motivation for this is twofold. First, understanding the behavior of this model in these limits will provide us with a whole slew of potential checks to further strengthen the identification between the NCCP^1 model and the hedgehog suppressed O(3) model. Some of these explicit checks, like the correspondence in the easy plane limits and the effect of a Zeeman field on the isotropic models, will be presented elsewhere,[25] while others are left for future work. Second, we will see below that these models have several interesting properties and may themselves be directly realizable. For example, the continuous transition seen in the numerical experiments of Ref. 20 has been conjectured in Ref. 19 to be described by the easy plane NCCP^1 model.

A. Easy plane deformation

We first consider modeling the easy plane deformation of our O(3) invariant sigma model with hedgehog suppression. This can be accomplished, e.g., by having ferromagnetic interaction between neighboring spins of the form $-J(n_i^x n_j^x + n_i^y n_j^y)$. Clearly, in this case the spins would prefer to lie in the x-y plane in spin space; the global spin symmetry of this model is now broken down from O(3) to $O(2)\times Z_2$, where the $O(2)$ corresponds to spin rotations about the z axis in spin space, and Z_2 arises from symmetry under $n^z\to -n^z$. The appropriate deformation of the NCCP^1 model involves applying the term $U(|z_1|^4+|z_2|^4)$ at each site with $U>0$ that favors equal amplitudes for the two components of the spinor field. The corresponding **n** vector will then lie in the easy plane. We will call this the easy plane NCCP^1 model.

In fact it will be useful to consider the limit of extreme easy plane anisotropy, where the amplitude fluctuations of the **z** fields are frozen out, and only the phase fluctuations remain. The **z** field can then be parametrized by $z_{1(2)} = e^{i\phi_{1(2)}}/\sqrt{2}$, and the direction of the spin in the easy plane is given by $n^x+in^y=2z_1^* z_2 = e^{i(\phi_2-\phi_1)}$. The partition function on a lattice in three Euclidean dimensions can then be written as

$$\mathcal{Z}_{EP} = \int_{-\infty}^{\infty}[D\mathbf{a}]' \int_{-\pi}^{\pi}[D\phi_1 D\phi_2] e^{-S_{EP}},$$

$$S_{EP} = -J\sum_{r,\mu}[\cos(\Delta_\mu\phi_1 - a_\mu) + \cos(\Delta_\mu\phi_2 - a_\mu)] + \frac{K}{2}\sum_{\square}(\Delta\times\mathbf{a})^2, \tag{15}$$

where the integration over the gauge fields is performed after suitable gauge fixing to ensure finiteness of the partition function (hence the prime in the integration measure: $[D\mathbf{a}]'$). Below, we will consider applying the standard duality transformations on this model.[26,27] For this purpose it is more convenient to pass to the Villain representation which makes use of the approximate rewriting $e^{J\cos\alpha}\approx\sum_n e^{in\alpha}e^{-n^2/2\mathcal{J}}$. Note that the Villain form has the same 2π periodicity $\alpha \to \alpha+2\pi$ as the original partition weight, and for large J we will have $\mathcal{J}\approx J$. The Villain form is expected to retain the universal properties and phase structure of the original model and therefore we will start with it and consider a series of exact transformations that perform the duality. The partition function written in the dual variables will be seen to be essentially the same as that written in the direct variables thus establishing the "self-dual" nature of these models.

The Villain form of the partition function is

$$Z_V = \int_{-\infty}^{\infty}[D\mathbf{a}]' \int_{-\pi}^{\pi}[D\phi_1 D\phi_2]\sum_{\mathbf{j}_1,\mathbf{j}_2} e^{-(K/2)\Sigma(\Delta\times\mathbf{a})^2} \times e^{-(1/2\mathcal{J})\Sigma(\mathbf{j}_1^2+\mathbf{j}_2^2)} e^{i\Sigma[(\Delta\phi_1-\mathbf{a})\mathbf{j}_1+(\Delta\phi_2-\mathbf{a})\mathbf{j}_2]}, \tag{16}$$

where $\mathbf{j}_{1,2}$ are integer current fields that live on the links of the lattice. We first write $\mathcal{Z}_V$ as a model in terms of the current loops only, and then perform an exact rewriting (duality) in terms of the vortex loops. The two forms will have essentially identical characters when viewed as integer loop models; with proper identification of the coupling constants, this is the statement of self-duality.

We begin by integrating out the phase fields $\phi_{1,2}$. This imposes the condition that the integer current fields $\mathbf{j}_{1,2}$ are divergence free. These integer loops are simply the world lines of the two bosons $\mathbf{z}_{1,2}$, which carry the same gauge charge and hence their sum couples to the gauge field. Integrating out the gauge field gives rise to a long range Biot-Savart-type interaction between the total currents. The result can be written as

$$\mathcal{Z}_V = \sum_{\mathbf{j}_1,\mathbf{j}_2}\delta(\Delta\cdot\mathbf{j}_1)\delta(\Delta\cdot\mathbf{j}_2)e^{-(1/2)(\mathbf{j}_1+\mathbf{j}_2)_r\mathcal{G}_+(r,r')(\mathbf{j}_1+\mathbf{j}_2)_{r'}} \times e^{-\frac{1}{2}(\mathbf{j}_1-\mathbf{j}_2)_r\mathcal{G}_-(r,r')(\mathbf{j}_1-\mathbf{j}_2)_{r'}} \tag{17}$$

$$\tilde{\mathcal{G}}_+(k) = \frac{1}{2\mathcal{J}} + \frac{1}{4K\left(\sin^2\frac{k_x}{2}+\sin^2\frac{k_y}{2}+\sin^2\frac{k_z}{2}\right)}, \tag{18}$$

$$\tilde{\mathcal{G}}_{-}(k) = \frac{1}{2\mathcal{J}}. \tag{19}$$

Thus, the original model is equivalent to a model of integer current loops,[28] where the combinations $\mathbf{j}_1 - \mathbf{j}_2$ have short-range interactions, while the gauge charged current combinations $\mathbf{j}_1 + \mathbf{j}_2$ have long-range interactions with the asymptotic form $\mathcal{G}_{+}(R \to \infty) = (1/4\pi K)(1/R)$.

We now perform an exact duality transformation on Eq. (17). First, the divergence free current loops can be written as the lattice curl of a vector field $\mathbf{j}_{1,2} = \Delta \times \mathbf{A}_{1,2}/2\pi$ that lives on the links of the dual lattice, and the integer constraint is implemented using the identity $(2\pi)^3 \sum_{\mathbf{n}} \delta(\mathbf{A} - 2\pi\mathbf{n}) = \sum_{\mathbf{l}} e^{i\mathbf{l}\cdot\mathbf{A}}$. Integrating out the fields $\mathbf{A}_{1,2}$ we obtain the following form for the partition function in terms of the dual integer current loops $\mathbf{l}_{1,2}$:

$$\mathcal{Z}_V \propto \sum_{\mathbf{l}_1,\mathbf{l}_2} \delta(\Delta \cdot \mathbf{l}_1)\delta(\Delta \cdot \mathbf{l}_2) e^{-(1/2)(\mathbf{l}_1 + \mathbf{l}_2)_r \mathcal{F}_{+}(r,r')(\mathbf{l}_1 + \mathbf{l}_2)_{r'}} \times e^{-(1/2)(\mathbf{l}_1 - \mathbf{l}_2)_r \mathcal{F}_{-}(r,r')(\mathbf{l}_1 - \mathbf{l}_2)_{r'}}, \tag{20}$$

$$\tilde{\mathcal{F}}_{+}(k) = \frac{\pi^2 K \mathcal{J}}{\mathcal{J} + 2K\left(\sin^2\frac{k_x}{2} + \sin^2\frac{k_y}{2} + \sin^2\frac{k_z}{2}\right)}, \tag{21}$$

$$\tilde{\mathcal{F}}_{-}(k) = \frac{\pi^2 \mathcal{J}}{2\left(\sin^2\frac{k_x}{2} + \sin^2\frac{k_y}{2} + \sin^2\frac{k_z}{2}\right)}. \tag{22}$$

One can argue that the two integer loops of this dual representation (20) are precisely interpreted as the vorticities in the original two boson fields. A vortex of equal strength in both boson fields ($\mathbf{l}_1 = \mathbf{l}_2$) can be screened by the gauge field, and hence has only short-range interactions, while unbalanced vortices which cannot be completely screened interact via long-range interactions. Correspondingly, the combination of the dual currents $\mathbf{l}_1 + \mathbf{l}_2$ interact via a short-range interaction $\mathcal{F}_{+}(R)$, while the other combination $\mathbf{l}_1 - \mathbf{l}_2$ has a long-range interaction with the same asymptotic form as in Eq. (18), $\mathcal{F}_{-}(R \to \infty) = (\pi\mathcal{J}/2)(1/R)$. Indeed the partition function written in terms of dual loops (20) is essentially the same as when written in terms of the direct variables (17), which can be seen by making the association $\mathbf{l}_1 \leftrightarrow \mathbf{j}_1$ and $\mathbf{l}_2 \leftrightarrow -\mathbf{j}_2$. The only differences are in the form of the short-range interactions which do not affect universal properties. As argued below this immediately implies that the transition between P^* and F phases in this model will be self-dual. Of course, we can also rewrite Eq. (20) in terms of the fields conjugate to the currents $\mathbf{l}_{1,2}$, obtaining the action in terms of the dual (vortex) fields and the dual gauge field, which will have essentially the same form as the original action Eq. (16). We do not spell this out here since the exhibited forms already suffice.

We first note the description of the various phases in terms of the properties of boson current loops and also the vortex (dual) loops. In the direct representation, when the partition function is dominated by small loops of $\mathbf{j}_1 \pm \mathbf{j}_2$, the system is in the "insulating" or paramagnetic state P^*. In this phase there is a single low-energy excitation—the photon. On the other hand, when these loops become arbitrarily large, which certainly occurs if both $J, K \gg 1$, we are in the "superfluid" or ferromagnetic phase F. Here too we have a single low-lying mode, the magnon, which is the Goldstone mode arising from the spontaneously broken spin symmetry within the easy plane.

The two integer loops of the dual representation (20) correspond to the world lines of the vortices in the two boson fields. It is easily seen, for example by analyzing the dual action in different parameter extremes, that the P^* phase occurs if *large* vortex loops of both $\mathbf{l}_1 \pm \mathbf{l}_2$ proliferate, while if both the vortex loops are typically small, the F phase results.

In terms of the direct boson variables, the F phase is the ordered phase, and hence has a low-energy Goldstone mode, while the P^* phase is the disordered phase and has a low-energy photon mode. In the dual variables the roles are reversed—the F phase is the disordered phase with the dual photon, while the P^* phase is the dual "ordered" phase, with the Goldstone mode. In particular, a direct transition between P^* and F can either be thought of as an ordering transition in the direct variables, or the reverse in terms of the dual variables. This interchange of ordered and disordered phases is typical of a duality transformation. What is special in this case is that the partition function is essentially identical when written in terms of the direct and dual variables. This implies that the transition must be self-dual. One direct consequence of this self duality is that the critical exponent (β) measuring the rise of the order parameter on the ordered side of the transition is equal to that of a suitably defined disorder parameter on the opposite side of the transition. Further consequences arising from the self-duality and other phases contained in this easy plane NCCP^1 model will be discussed at length elsewhere,[25] along with results of Monte Carlo studies. Here, we just note that a continuous transition from the P^* to F phase is obtained, with critical indices

$$\nu = 0.60 \pm 0.05, \quad \beta/\nu = 0.70 \pm 0.05 \ \ (\text{easy plane}). \tag{23}$$

Finally, we note that this critical point is conjectured in Ref. 19 to control the continuous transition seen in the numerical experiments of Ref. 20.

B. Effect of Zeeman field

We now consider applying a uniform Zeeman field along the n^z direction in spin space, $S_Z = -h\sum_i n_i^z$, and study the effect on the various phases and phase transitions in both the isotropic and easy plane models. In the CP^1 representation this term can be written as

$$S_Z = -h \sum (|z_1|^2 - |z_2|^2) \tag{24}$$

which breaks the $z_1 \leftrightarrow z_2$ and the only remaining symmetry is that of U(1) spin rotations about the n^z axis.

For the model with the isotropic exchange coupling, adding the term (24) to the action (5) will result in the phase diagram shown in Fig. 7(a). At zero field we have the P^* and F phases separated by the monopole suppressed O(3) transition, or equivalently the NC CP^1 ordering transition C_I.

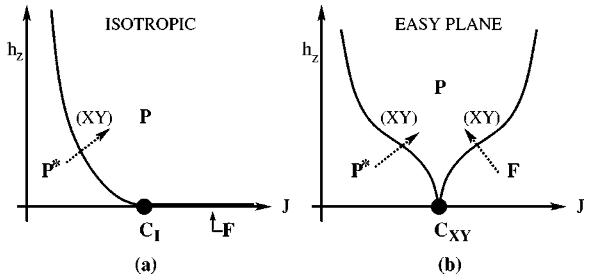

FIG. 7. Effect of a Zeeman field on the zero-temperature phases of the NCCP^1 model. (a) Isotropic case—two distinct paramagnetic phases exist in the presence of a Zeeman field. The P^* phase has a low lying excitation, the photon, while the P phase is gapped and breaks no global symmetries. The two are separated by a continuous transition in the 3D XY universality class (the arrow indicates the ordered to disordered 3D XY transition). At zero field the transition C_I is described by the critical NCCP^1 model. (b) Easy plane case—Zeeman field is applied perpendicular to the easy plane. Spins can order in the easy plane to give the F phase which has a low lying excitation, the Goldstone mode. This is separated from the disordered P phase by a 3D XY transition. There are two paramagnetic phases, P^* with a photon and a gapped P phase separated by a 3D XY transition. At zero Zeeman field the two 3D XY transitions meet to give the self dual C_{XY} transition.

Turning up the Zeeman field, the ordered moment of F locks into the field direction—the resulting phase does not break any of the global symmetries and has no low-lying modes. Hence we will call this phase P. (Note that for the isotropic NCCP^1 model at zero temperature, this phase is only present when a Zeeman field is applied.) This phase can be understood as a Higgs phase with condensation of the z_1 field but not of the z_2 field.

On the other hand, in the P^* phase there is a gap to z_1 and z_2 particles. Turning up the Zeeman coupling lowers the z_1 branch until at the transition it condenses to give the P phase. Thus, the transition between the P^* and P phases involves condensing a single scalar field coupled to a gauge field and hence is expected to be in the inverted-XY universality class[26] (i.e., the P^* to P transition is the ordered phase to disordered phase transition of the 3D XY model).

The shape of the phase boundary near the zero field transition can be related to the critical exponents of the NC CP^1 model. In particular it depends on the ratio of the scaling dimension of the Zeeman operator ($y_h = d-\beta/\nu$ for the isotropic case) to the scaling dimension of the "thermal" operator ($y_t = 1/\nu$). Thus, if δJ is the deviation away from the zero field critical point C_I, then the phase boundary is given by the curve

$$h_z \propto (\delta J)^{\phi}, \tag{25}$$

$$\phi = y_h/y_t. \tag{26}$$

The expressions for the scaling dimensions yield $\phi = y_h/y_t = \nu d - \beta$. Using the numerical values of the exponents in Eq. (14), we have $\phi \approx 1.7$, which gives a phase boundary that approaches the isotropic transition point horizontally as shown in Fig. 7(a).

For the model with easy plane exchange coupling, the effect of adding the Zeeman field perpendicular to the easy plane can be argued to lead to the phase diagram in Fig. 7(b). Here we avoid using the extreme easy plane limit, where n^z was set identically to zero, in order to have a finite coupling to the applied field. [Alternatively, we can model the effect of the Zeeman field by considering an action similar to Eq. (16) but with different couplings J_1 and J_2 for the two angle variables.] At zero field we have the P^* and F phases separated by the monopole suppressed easy plane transition. Again, in the presence of a Zeeman field, a P phase is possible, with no spontaneously broken symmetry and no low lying excitations, where only the z_1 field is condensed. The phase transition between P^* and P is then driven by the condensation of the gauge charged z_1 field, and is hence expected to be in the same universality class as the ordered to disordered transition of the 3D XY model.[26] Starting in the P phase, as the exchange coupling is further increased, it becomes favourable for the z_2 field to also condense, so that there develops a finite expectation value for the in-plane spin operators $n^x + i n^y = 2 z_1^* z_2$. This leads to the F phase, which spontaneously breaks the remaining spin rotation symmetry and has one Goldstone mode. Since the gauge field has already been "Higgsed" away, this transition is the regular ordering transition of the 3D XY model. As described previously, the original easy plane model is essentially self-dual, and this remains true on adding the Zeeman term. This explains the "reflection" symmetry of the universal properties in the phase diagram—for example, the duality interchanges the P^* and F phases while the P phase is taken to itself. This also implies that the transition from P^* to P must be in the same universality class as the transition from F to P, which was indeed the result of the previous analysis, which finds them both to be in the 3D XY universality class.

The shape of the phase boundaries near the zero field easy plane transition can be related to the critical exponents of the easy plane NC CP^1 model using Eq. (26). While the thermal eigenvalue in this case is already known [Eq. (23)], the scaling dimension of the Zeeman operator needs to be determined. This is conveniently done within the easy plane NC CP^1 model at criticality, by studying the scaling dimension of the operator that gives rise to unequal hopping strengths $J_1 \neq J_2$ for the two species of bosons in Eq. (16). To this end, consider defining the link operators

$$O_{ij}^{\pm}(r) = \{\cos(\Delta\phi_1 - a) \pm \cos(\Delta\phi_2 - a)\}. \tag{27}$$

Clearly, adding $\Sigma_{\langle ij\rangle} O_{ij}^-$ will give rise to unequal hoppings, and hence has an overlap with the Zeeman operator. Similarly, $\Sigma_{\langle ij\rangle} O_{ij}^+$ will have an overlap with the "thermal" operator. Thus, studying the power law decay of the correlators of $O_{ij}^{\pm}$ at criticality allows us to extract the scaling dimensions of the Zeeman and thermal operators. The latter may be compared against other, more accurate determinations of the same quantity and serves as a check of this approach. Thus we find

$$y_h = 1.2 \pm 0.3, \tag{28}$$

$$y_t = 1.6 \pm 0.2 \tag{29}$$

the value of y_t obtained is consistent with the value of ν quoted in Eq. (23) from which $y_t \approx 1.7$. Thus we have $\phi = 0.75 \pm 0.2$. Note that since $0 < \phi < 1$, the shapes of the phase boundaries are as shown in Fig. 7(b).[29]

Finally we note that these properties of the easy plane model in the presence of a perpendicular Zeeman field are potentially useful to testing the conjecture of Ref. 19 that the continuous transition seen in the numerical experiments of Ref. 20 is controlled by the critical easy plane NC CP^1 model. An important point is that given the mapping of the Néel field to the unit vector of the sigma model, the Zeeman field considered here actually corresponds to a staggered Zeeman field on the spins of Ref. 20.

C. Finite temperatures

We now investigate the finite temperature properties of the NC CP^1 model [or equivalently, the monopole suppressed O3 sigma model], both for the isotropic as well as the easy plane case. The main result is that there is a finite temperature version of the P^* phase, which we call the thermal Coulomb phase P_T^*, with power law correlations of the electric fields. This is distinct from the regular paramagnetic phase P with short-ranged correlations, and is separated from it by a finite temperature Kosterlitz-Thouless phase transition. The existence of such a transition can be seen from the following physical argument. In the zero-temperature P^* phase, there are gapped spinons that interact via a logarithmic interaction. If this interaction persists to finite temperatures, then clearly for small enough temperatures, thermal fluctuations will only generate neutral spinon pairs. This is the P_T^* phase. However, at some higher temperature it becomes entropically favorable to proliferate unbound spinons (exactly as with logarithmically interacting vortices in the two-dimensional XY model) and a Kosterlitz-Thouless type transition to a spinon-plasma phase will be expected. Indeed, for the hedgehog suppressed O(3) model, the existence of such a finite-temperature transition may be viewed as evidence for the existence of emergent gauge charged objects interacting via a Coulomb potential, that happens to be logarithmic in two spatial dimensions which gives rise to the transition.

We now sketch a derivation of these results—we choose to do this in a model with a single species of "spinon" or gauge charged particle. Treating this single component (or noncompact CP^0) model will simplify the discussion and the very same results hold for the models of interest, i.e., the isotropic and easy-plane NCCP^1 models, because it is only the logarithmic binding/unbinding of the gauge charged objects that matters. Since the finite temperature physics described above occurs in the charge and electric field sector, it will be useful to consider writing the thermal partition function $\mathcal{Z}_\beta = \mathrm{Tr}\, e^{-\beta H}$ in terms of the zero Matsubara frequency components of these fields. The appropriate effective energy is

$$E\{n_r, e_{r\mu}\} = \frac{U}{2}\sum_r n_r^2 + \frac{1}{2\mathcal{K}}\sum_{r\mu} e_{r\mu}^2, \tag{30}$$

where r and μ run over the sites and directions of the two-dimensional spatial lattice. The partition function involves a sum over all configurations of the integer charge fields n_r and real electric fields $e_{r\mu}$ (since we are working with a noncompact gauge theory) that satisfy the Gauss law constraint $\Delta \cdot \mathbf{e} = n_r$ at each site of the lattice. Note that the boson number and electric field variables that we use here are conjugate to the phase and gauge potential variables that have been used so far [see Eqs. (5) and (6)]. Our treatment here is precise in the regime where the total Hamiltonian is dominated by the exhibited $E\{n_r, e_{r\mu}\}$ part, but is also valid more generally in the effective sense.

The Gauss law constraint can be implemented using a Lagrange multiplier a_0 to give the following expression for the partition function:

$$\mathcal{Z}_\beta \sim \sum_{\{n_r\}} \int [D\mathbf{e}] \int [Da_0] e^{-\beta E\{n_r, e_{r\mu}\} + i\Sigma a_0(\Delta\cdot\mathbf{e} - n)}. \tag{31}$$

To establish that there is a thermal transition, we integrate out the fields $a_0(r), e_{r\mu}$, so the partition weight is now just a function of the integer charges. This yields

$$\mathcal{Z}_\beta \sim \sum_{\{n_r\}} e^{-(\beta/2)\Sigma_{rr'} n_r V_{rr'} n_{r'}}, \tag{32}$$

where for large separation the potential between charges takes the form $V_{rr'} \sim (1/2\pi\mathcal{K})\log(1/|r-r'|)$, which is the Coulomb interaction in two dimensions. It is well known that such a Coulomb gas has two phases, one with the charges bound into neutral pairs at low temperatures, and a plasma phase at high temperatures separated by a KT transition which occurs at $T_c \leqslant (4\pi\mathcal{K})^{-1}$. This is not surprising since we know that NC CP^0 model is dual to the O(2) quantum rotor model in two spatial dimensions, and the spinons of the NC CP^0 map onto the vortices of the O(2) rotor model, which interact logarithmically.

A sharp distinction between the low and high temperature paramagnetic phases can be drawn by looking at the electric field correlators. In the low- (but nonzero) temperature P_T^* phase, these fall off as the inverse square of the distance, while in the high-temperature "spinon plasma" phase P, these correlators are short ranged:

$$\langle e_\mu(r) e_\nu(0)\rangle \sim \begin{cases} (1/r^2)(2r_\mu r_\nu/r^2 - \delta_{\mu\nu}) & T < T_c, \\ \text{short ranged} & T > T_c, \end{cases} \tag{33}$$

where the indices $\mu, \nu \in \{x, y\}$. As shown elsewhere,[25] exactly the same results are obtained for the isotropic NC CP^1 model, and the phase diagram is shown in Fig. 8(a). In particular, despite the presence of a global SU(2) symmetry, the transition remains KT.

For the easy plane NCCP^1 model, the expected finite-temperature phase diagram is shown in Fig. 8(b). Here, in addition to the discussed P_T^* and P phases, there is also a power law phase (XY) that appears out of the zero-temperature ferromagnetically ordered phase and terminates at the usual KT transition to the P phase. Again, the similarity between the P_T^* and XY sides of this phase diagram have to do with the self duality of the underlying easy-plane NC CP^1 model.

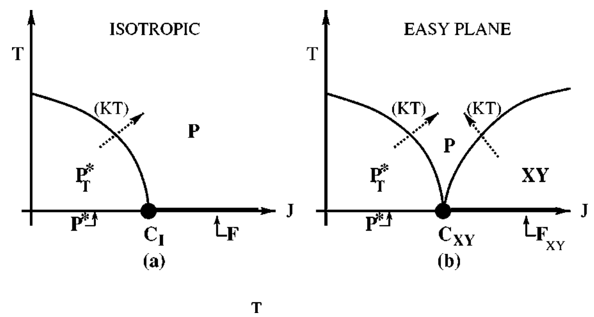

FIG. 8. Effect of finite temperature T on the NC CP^1 model. (a) Isotropic case: The thermal Coulomb phase P_T^*, a finite temperature analog of P^*, has power law electric field correlations [Eq. (33)] and is separated from the more conventional P phase by a Kosterlitz-Thouless phase transition. This can be understood as the thermal ionization of spinons interacting via the 2D Coulomb (log) potential. The F phase is disordered at any finite temperature. (b) Easy plane case: The ordered phase now has algebraic correlations at finite temperature (XY) which is also separated from the P phase by a KT transition.

IV. TOWARDS $D=3+1$ AND LAYERED PHASES

All of the preceding discussion was focused on the $(2+1)$-dimensional system where complete hedgehog suppression was required in order to obtain the deconfined phase. It is natural to ask whether similar physics can be obtained in models with only a finite energy cost for hedgehogs. This can occur in the presence of critical bosonic fields as argued in Ref. 19 leading to deconfined quantum criticality in $(2+1)$-dimensional systems. Another way in which a deconfined phase can be stabilized with a finite hedgehog fugacity is to consider $(3+1)$-dimensional systems. Now, the hedgehog is a particle, and a finite hedgehog fugacity does not necessarily destabilize the Coulomb phase. Thus, in principle one could obtain a $(3+1)$-U(1) deconfined phase with global SU(2) symmetry, by disordering the O(3) sigma model with a sufficiently large but finite hedgehog core energy. This question is currently under investigation; the obvious extensions of the presented $(2+1)$-dimensional realization either led to ferromagnetic order or to the conventional paramagnet.

An interesting possibility that we remark upon is a layered system that lives in three spatial dimensions, where each layer realizes the $(2+1)$-dimensional deconfined phase. Such a phase exhibits *quantum confinement* in that the gauge charged spinons can move freely within a layer but not between the layers, and the photon remains a $(2+1)$-dimensional particle. This is reminiscent of the sliding Luttinger liquid phase[30] of coupled one-dimensional systems that also exhibits the phenomenon of quantum confinement, and has a low-energy phonon excitation (which is the analogue of the photon in this lower-dimensional system).

V. CONCLUSIONS

We conclude by highlighting the main results of this work. We considered the question of whether the O(3) sigma model can be disordered without monopoles, and answered it in affirmative. We provided an explicit example of an O(3) Heisenberg spin system with no free hedgehogs and no magnetic order in $(2+1)D$. While this is in agreement with the earlier work of Kamal and Murthy,[14] the potentially problematic features of the procedure adopted in that work have been entirely avoided in the present paper.

Furthermore, we identified the proper description of the hedgehog suppressed O(3) model which involves spinons coupled to a noncompact gauge field, which may be called the noncompact CP^1 model. Thus the hidden topological order resulting from hedgehog suppression gives rise to a photonlike low-energy excitation in the paramagnetic (P^*) phase of this model, which leads to power law correlations of the spin chirality. This may also be viewed as an example of a U(1) fractionalized phase (albeit with complete monopole suppression) with full SU(2) spin rotation symmetry.

Our understanding of the hedgehog suppressed disordered sigma model is interesting from the statistical mechanics point of view and addresses some long standing questions. In a sense, we identified how to "decompose" the O(3) model into a part that involves the topological defects (hedgehogs), and the part which does not involve these. Such decomposition of the O(2) model into the vortex and spin wave parts is well known. The corresponding "spin wave" part for the O(3) model turns out to be the NC CP^1 model, which clearly has little to do with spin waves. In particular, the commonly asked question whether the spin waves (perhaps nonlinearly coupled) can disorder the O(3) model seems to have no meaning.

Another question has been on the role played by the hedgehog defects at the Heisenberg transition. A sharp formulation of this question is whether the ordering transition in the hedgehog suppressed model is identical to the Heisenberg transition. Our calculations of the universal critical exponents for this transition ($\nu=1.0\pm0.2$ and $\beta/\nu=0.80\pm0.05$) show that it is indeed distinct from the Heisenberg transition. Direct simulation of the NCCP^1 model, however, yields exponents that are consistent with these. Moreover, the large $\eta\approx0.6$ of the vector (magnon) field implied by these exponents can be heuristically understood since the magnon can decay into a pair of unconfined spinons at this critical point.

Thus, an important conceptual issue that is clarified by our work is the existence of two different spin rotation symmetric critical points in $D=2+1$. The first is the Heisenberg transition whose "soft spin" field theory is given by the O(3) ϕ^4 theory. This describes, of course, the transition in the O(3) sigma model if it is regularized by putting the spins on the lattice. It also describes the transition in the lattice CP^1 model with a compact gauge field. The second transition is described by a "soft spin" field theory with a pair of complex scalar fields (that transform as spinors under spin rotations), coupled to a noncompact gauge field. This describes the ordering transition of the lattice O(3) sigma model with hedgehog suppression. It also describes the transition in the lattice CP^1 coupled to a noncompact gauge field. Earlier indiscriminate use of what is effectively the NC CP^1model to describe the Heisenberg transition are therefore to be reconsidered.

We also studied various physical extensions of the hedgehog suppressed O(3) model, such as adding Zeeman field and also the effect of finite temperature. The P^* phase survives in small external magnetic field and retains the photon, which is an evidence for gapped spinons. At finite temperatures, a thermal Coulomb phase with long-range power law electric field correlations exists, and undergoes a Kosterlitz-Thouless transition to the usual paramagnetic phase at higher temperatures; this is indirect evidence for gauge charge carrying particles (spinons) that interact via the Coulomb potential which is logarithmic in 2D.

An interesting extension is obtained by breaking the full rotation symmetry down to the easy-plane XY symmetry. We showed that this model possesses a remarkable "self-duality" property. Indeed as far as we know this is the first purely bosonic model in $D=2+1$ that displays this property. In particular the critical point describing the ordering transition in this model is self-dual. An important role is played by this critical theory in a forthcoming publication,[19] where it is conjectured that this critical point may already have been seen in numerical experiments on easy plane spin half quantum antiferromagnet on the square lattice where surprisingly a continuous transition is observed between a spin ordered state and a valence bond solid that breaks lattice symmetries.[20]

In $D=2+1$ the P^* phase and its gapless photon excitation only exist in the limit of infinite hedgehog suppression. Thus, within this phase, the complete suppression of free hedgehogs represents an extreme limit that may seem unnatural. However, reasoning in this limit can be conceptually powerful and throw light on many tricky issues. In addition, we note that even if hedgehogs eventually proliferate and lead to a gap for the photon, this P^* phase may be relevant for describing physical crossovers in some strongly correlated systems at energy scales above the photon mass. In contrast to the P^* phase which in the CP^1 language is unstable to turning on a finite monopole fugacity, the stability of the critical points where gapless gauge charged particles are present is a more involved question, and depends on the number of gapless fields present and the particular monopole creation operator under consideration. In fact, it is argued in Ref. 19 that quadrupled monopole operators (which are relevant to the spin half quantum antiferromagnets on the square lattice) are irrelevant at the NCCP^1 critical point in both the isotropic and easy plane limits. This would allow for a continuous transitions between valence bond solid and Néel phases in such systems, and these transitions would be controlled by the critical points studied in this paper. Finally, we expect a similar P^* phase in the $(3+1)D$ O(3) model, which will now be present also with strong but finite monopole suppression, which would provide a spin rotation invariant $(3+1)D$ model which exhibits a fractionalized Coulomb phase with deconfined spinons and a true photon excitation.

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